Int. J. Heat Mass Transfer. Vol. 33, No. 2, pp. 225–228, 1990
Printed in Great Britain

0017-9310/90 \$3.00+0.00
Pergamon Press plc

Some gas dynamic relations for real gas flows in the presence of heat transfer

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(*Received* 25 *February* 1985 *and in final form* 11 *May* 1988)

Abstract—Based on thermodynamic differential equations some relations are derived which take into account the nonideality of the gas, energy transfer and friction.

IN AN EARLIER paper [1] the present author suggested the following relation:

$$\mathrm{d}\frac{w^2}{2}+\left(\frac{\partial f}{\partial p}\right)_T\mathrm{d}p+R\frac{k_T}{k_T-1}\eta\left(\mathrm{d}T-\frac{\mathrm{d}Q}{C_p}\right)+\mathrm{d}L_{\mathrm{tech}}+\mathrm{d}L_{\mathrm{fr}}=0. \quad (1)$$

The quantities η and $k_T/(k_T-1)$ vary very little in the course of gas motion, therefore their mean values, $\bar{\eta}$ and $k_T/(k_T-1)$, will be employed to integrate relation (1) (for convenience the bar over the latter quantity is omitted). Thus, integration will yield

$$\frac{w_2^2-w_1^2}{2}+\int_1^2\left(\frac{\partial f}{\partial p}\right)_T\mathrm{d}p+R\frac{k_T}{k_T-1}\bar{\eta}(T_2-T_1)-\frac{\bar{\eta}}{\bar{\omega}}\int_1^2\mathrm{d}Q+L_{\mathrm{tech}}+L_{\mathrm{fr}}=0 \quad (2)$$

$$\mathrm{d}Q=\mathrm{d}Q_{\mathrm{ex}}+\mathrm{d}Q_{\mathrm{fr}} \quad (3)$$

where $\mathrm{d}Q_{\mathrm{ex}}$ is the external heat supplied, and $\mathrm{d}Q_{\mathrm{fr}}$ the heat of frictional work. The power of the supplied heat is $W_{\mathrm{ex}}=\mathrm{d}Q_{\mathrm{ex}}/\mathrm{d}t$, and the power of the friction forces is $W_{\mathrm{fr}}=\mathrm{d}Q_{\mathrm{fr}}/\mathrm{d}t=\mathrm{d}L_{\mathrm{fr}}/\mathrm{d}t$, $W_{\mathrm{tech}}=\mathrm{d}L_{\mathrm{tech}}/\mathrm{d}t$.

In the absence of technical work, i.e. at $\mathrm{d}L_{\mathrm{tech}}=0$, the energy equation has the form

$$\frac{\bar{\eta}}{\bar{\omega}}\frac{W_{\mathrm{ex}}}{G}=\frac{w_2^2-w_1^2}{2}+\int_1^2\left(\frac{\partial f}{\partial p}\right)_T\mathrm{d}p+R\frac{k_T}{k_T-1}\bar{\eta}(T_2-T_1). \quad (4)$$

When deriving the above equation, all the transformations were made with a view to obtain relations in terms of p and T and, generally, in what follows to use the temperature ratio of specific heats k_T for real gases. An insignificant variation of $(k_T-1)/k_T$ simplifies manipulations and ensures a sufficient accuracy of integration.

Now, the integral of equation (4), i.e.

$$\int_1^2\left(\frac{\partial f}{\partial p}\right)_T\mathrm{d}p$$

will be expanded. Introducing the notation

$$x=pu=f(p,T)$$

it is possible to write

$$\mathrm{d}x=\left(\frac{\partial x}{\partial p}\right)_T\mathrm{d}p+\left(\frac{\partial x}{\partial T}\right)_p\mathrm{d}T$$

then

$$\int_1^2\left[\frac{\partial(pu)}{\partial p}\right]_T\mathrm{d}p=\int_1^2\left(\frac{\partial x}{\partial p}\right)_T\mathrm{d}p=\int_1^2\mathrm{d}x-\int_1^2\left(\frac{\partial x}{\partial T}\right)_p\mathrm{d}T=x_2-x_1-\int\left(\frac{\partial x}{\partial T}\right)_p\mathrm{d}T$$

or

$$\int_1^2\left[\frac{\partial(pu)}{\partial p}\right]_T\mathrm{d}p=p_2u_2-p_1u_1-\int_1^2\left[\frac{\partial(pu)}{\partial T}\right]_p\mathrm{d}T. \quad (5)$$

A number of transformations will give

$$\int_1^2\left(\frac{\partial f}{\partial p}\right)_T\mathrm{d}p=RT_1[(Z_2)_{T_1}-(Z_1)_{T_1}] \quad (6)$$

where $(Z_1)_{T_1}$ is taken at pressure p_2 and at the initial temperature T_1 [2].

Substituting equation (6) into equation (2) yields

$$\frac{w_2^2-w_1^2}{2}+R\frac{k_T}{k_T-1}\bar{\eta}(T_2-T_1)+RT_1[(Z_2)_{T_1}-(Z_1)_{T_1}]-\frac{\bar{\eta}}{\bar{\omega}}\int_1^2\mathrm{d}Q+L_{\mathrm{tech}}+L_{\mathrm{fr}}=0. \quad (7)$$

This is a very important relation representing the energy equation of a real gas.

Often, when the problems of ideal gas flow in the absence of heat transfer are considered, the following relation [3] is used:

$$\left(\frac{T}{T_0}\right)_{\mathrm{id}}=1-\frac{k-1}{k+1}\lambda_{\mathrm{id}}^2. \quad (8)$$

It is known that in an energetically isolated gas flow the stagnation temperature is a constant quantity. In the presence of heat transfer the quantity T_0 in equa-

NOMENCLATURE

a speed of sound in real gas [m s^{-1}]
a_{cr} critical velocity of real gas [m s^{-1}]
a_{id} speed of sound in ideal gas [m s^{-1}]
$(a_{cr})_{id}$ critical velocity of ideal gas [m s^{-1}]
C_p isobaric heat capacity of real gas [J kg^{-1} K^{-1}]
F tube cross-section [m^2]
G gas flow rate [kg s^{-1}]
k specific heat ratio of ideal gas, C_p/C_v
k_T temperature index of real gas adiabatic, $[1-p/T(\partial T/\partial p)_s]^{-1}$
p pressure [N m^{-2}]
R gas constant [J kg^{-1} K^{-1}]
T temperature [K]
u specific volume [m^3 kg^{-1}]
w gas velocity [m s^{-1}]
y correcting factor, a/a_{id}
Z coefficient of compressibility.

Greek symbols
λ velocity factor
ξ correcting factor
ξ_{cr} $a_{cr}/(a_{cr})_{id}$
ρ density [kg m^{-3}].

Subscripts
cr critical
id ideal
s entropy.

tion (8) is variable. When higher accuracy is required, equation (8) is written as

$$\left(\frac{T}{T'_0}\right)_{id} = 1-\frac{k-1}{k+1}\lambda_{id}^2 \quad (9)$$

where $(T'_0)_{id}$ is the variable stagnation temperature. For real gases the relation becomes more complex. For this relation to be obtained, the following derivations should be made.

As a result of the flow adiabatic stagnation, the following replacements are to be made in equation (7):

$$T_2 = T_0; \quad p_2 = p_0; \quad T_1 = T; \quad p_1 = p$$
$$(Z_1)_{p_1,T} = (Z_2)_{p,T}; \quad (Z_2)_{p_2,T_1} = (Z)_{p_0,T}.$$

The stagnation flow velocity will decrease from $w_1 = w$ to $w_2 = 0$. Then, at $L = 0$, $L_{fr} = 0$ and $dQ = 0$

$$R\left\{T[(Z)_{p_0,T}-(Z)_{p,T}]+\bar{\eta}\frac{k_T}{k_T-1}(T_0-T)\right\} = \frac{w^2}{2}$$

or whence

$$\frac{T_0}{T} = 1+\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}\left\{\frac{1}{2}\frac{w^2}{RT}-[(Z)_{p_0,T}-(Z)_{p,T}]\right\}. \quad (10)$$

In an earlier paper [4] an expression was obtained for the equilibrium speed of sound in a real or a dissociating gas. For the real gas it has the form

$$a = Z\sqrt{\left(\frac{RT}{\eta-\frac{R}{C_p}\omega^2}\right)}. \quad (11)$$

The relationship between a and the speed of sound in an ideal gas will be expressed as

$$a = ya_{id} \quad (12)$$

where, as is known

$$a_{id} = \sqrt{(kRT)} \quad (13)$$

and y is a correcting factor.

Solving equations (11)–(13) simultaneously gives

$$y^2 = \frac{Z^2}{k\left(\eta-\frac{R}{C_p}\omega^2\right)}. \quad (14)$$

On having introduced the Mach number M $(=w/a)$ and taking into account the fact that $RT = a^2/ky^2$, it is possible to rewrite equation (10) in the form

$$\frac{T_{0x}}{T} = 1+\frac{1}{\eta}\frac{k_T-1}{k_T}\left\{\frac{k}{2}y^2M^2-[(Z)_{p_{0x},T}-(Z)_{p,T}]\right\}. \quad (15)$$

Since

$$\frac{T_0}{T} = \left(\frac{p_0}{p}\right)^{(k_T-1)/k_T}$$

it is possible to write

$$\frac{p_{0x}}{p} = \left[1+\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}\left\{\frac{k}{2}y^2M^2-[(Z)_{p_{0x},T}-(Z)_{p,T}]\right\}\right]^{k_T/(k_T-1)}. \quad (16)$$

The quantities T_{0x} and p_{0x} are, respectively, the variable stagnation temperature and pressure of a gas cooled while moving in a gas pipe-line.

Now, the stagnation parameters will be obtained and expressed in terms of the velocity λ.

By virtue of equations (12) and (13), the square of the critical velocity is

$$a_{cr}^2 = y_{cr}^2kRT_{cr} \quad (17)$$

or, since

$$(a_{cr}^2)_{id} = 2\frac{k}{k+1}RT_{0x} \quad (18)$$

it is possible to write

$$a_{cr}^2 = \xi_{cr}^2 a_{cr\,id}^2 = 2\xi_{cr}^2\frac{k}{k+1}RT_{0x}. \quad (19)$$

Equations (17) and (19) yield

$$\xi_{cr}^2 = \frac{k+1}{2}y_{cr}^2\frac{T_{cr}}{T_{0x}}. \quad (20)$$

When $M = 1$, T in equation (15) should be equal to T_{cr}. Having found the critical temperature from equation (15) and substituted it into equation (20), one obtains

$$\xi_{cr}^2 = y_{cr}^2\frac{k+1}{2}\left[1+\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}\left\{\frac{k}{2}y_{cr}^2 - [(Z)_{p_{0x},T_{cr}}-(Z)_{p_{cr},T_{cr}}]\right\}\right]^{-1}. \quad (21)$$

Since $Ma+\lambda a_{cr}$, equations (12), (13), and (19) yield

$$y^2M^2 = \xi_{cr}^2\frac{2}{k+1}\frac{T_{0x}}{T}\lambda^2.$$

The substitution of the latter relation into equation (15) gives

$$\frac{T_{0x}}{T} = \left\{1-\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}[(Z)_{p,T}-(Z)_{p_{0x},T}]\right\} \times\left(1-\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}\frac{k}{k+1}\xi_{cr}^2\lambda^2\right)^{-1}. \quad (22)$$

Correspondingly

$$\frac{p_{0x}}{p} = \left\{1+\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}[(Z)_{p,T}-(Z)_{p_{0x},T}]\right\}^{k_T/(k_T-1)} \times\left(1-\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}\frac{k}{k+1}\xi_{cr}^2\lambda^2\right)^{-k_T/(k_T-1)}. \quad (23)$$

Now, the relation will be derived which characterizes the change in gas pressure along a cylindrical pipe in the presence of heat transfer.

According to the equation

$$pu = ZRT \quad (24)$$

the variables in the process of heat removal are

$$p_{0x} = (Z)_{p_{0x},T_{0x}}\rho_{0x}RT_{0x} \quad (25)$$

and

$$p = (Z)_{p,T}\rho RT. \quad (26)$$

Dividing the latter equation by the former, with equation (21) taken into account, gives

$$\frac{p}{p_{0x}} = \frac{(Z)_{p,T}}{(Z)_{p_{0x},T_{0x}}}\frac{\rho}{\rho_{0x}}\left(1-\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}\frac{k}{k+1}\xi_{cr}^2\lambda^2\right) \times\left\{1+\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}[(Z)_{p,T}-(Z)_{p_{0x},T}]\right\}^{-1}. \quad (27)$$

The gas density is

$$\rho = \frac{G}{F\lambda\, a_{cr}}. \quad (28)$$

In the tube section under consideration

$$a_{cr}^2 = 2\xi_{cr}^2\frac{k}{k+1}RT_{0x}$$

or, according to equation (25)

$$a_{cr}^2 = 2\xi_{cr}^2\frac{k}{k+1}R\frac{p_{0x}}{\rho_{0x}(Z)_{p_{0x},T_{0x}}}.$$

Having determined from this equation p_{0x} and substituted it into equation (27) allowing for equation (28), it is possible to express the change in gas pressure along the pipe as

$$p = (Z)_{p,T}\frac{k+1}{2k}\frac{G}{F}\frac{1}{\xi_{cr}^2}a_{cr}\frac{1}{\lambda}\left(1-\frac{k_T-1}{k_T}\frac{k}{k+1}\frac{1}{\bar{\eta}}\xi_{cr}^2\lambda^2\right) \times\left\{1+\frac{1}{\bar{\eta}}\frac{k_T-1}{k_T}[(Z)_{p,T}-(Z)_{p_{0x},T}]\right\}^{-1}. \quad (29)$$

Another important relation will also be derived with the use of Bernoulli's equation at $L_{tech} = 0$.

Taking into account the fact that

$$dL_{fr} = \xi\frac{w^2}{2D}dl$$

where ξ is the friction coefficient in a pipe, dl the length of an elementary pipe section, and D the pipe diameter, the Bernoulli equation can be written in the form

$$\frac{dp}{\rho}+d\frac{w^2}{2}+\xi\frac{w^2}{2D}dl = 0$$

or

$$\frac{dp}{\rho w^2}+\frac{dw}{w}+\xi\frac{dl}{2D} = 0. \quad (30)$$

For differentiation, the quantities $(k_T-1)/k_T$, ξ_{cr} and η can be sufficiently accurately regarded constant because they change significantly in the course of gas expansion. Moreover, for the convenience of differentiation, equation (29) can be simplified taking into consideration that there is a very small difference within the square brackets, the more so that it is multiplied by the very small quantity $1/\bar{\eta}[(k_T-1)/k_T]$. In view of the above, equation (29) takes on the form

$$p = (\bar{Z})_{p,T}\frac{k+1}{2k}\frac{G}{F}\frac{1}{\xi_{cr}^2}a_{cr}\frac{1}{\lambda} \times\left(1-\frac{k_T-1}{k_T}\frac{k}{k+1}\frac{1}{\bar{\eta}}\xi_{cr}^2\lambda^2\right) \quad (31)$$

where

$$(\bar{Z})_{p,T} = \tfrac{1}{2}[(Z)_{p_1,T_1}+(Z)_{p_2,T_2}].$$

Differentiation of equation (31) yields

$$\mathrm{d}p = A\frac{G}{F}\left(\frac{k+1}{2k}\frac{\mathrm{d}a_{cr}}{\lambda}-\frac{k+1}{2k}a_{cr}\frac{\mathrm{d}\lambda}{\lambda^2} -\frac{1}{2}Ba_{cr}\,\mathrm{d}\lambda-\frac{1}{2}B\lambda\,\mathrm{d}a_{cr}\right) \quad (32)$$

where

$$A = (\bar{Z})_{p,T}\frac{1}{\xi^2}; \quad B = \frac{1}{\bar{\eta}}\xi_{cr}^2\frac{k_T-1}{k_T}.$$

As is shown

$$G = \rho wF. \quad (33)$$

Since

$$w = \lambda a_{cr} \quad (34)$$

$$\frac{\mathrm{d}w}{w} = \frac{\mathrm{d}a_{cr}}{a_{cr}}+\frac{\mathrm{d}\lambda}{\lambda}. \quad (35)$$

Simultaneous solution of equations (30) and (32)–(35) gives

$$\left(1-\frac{1}{\bar{\eta}}\xi_{cr}^2\frac{k_T-1}{2k_T}-\frac{k+1}{2k}\frac{1}{\lambda^2}\right)\frac{\mathrm{d}\lambda}{\lambda} +\left(1-\frac{1}{\bar{\eta}}\xi_{cr}^2\frac{k_T-1}{2k_T}+\frac{k+1}{2k}\frac{1}{\lambda^2}\right) \mathrm{d}\ln\frac{a_{cr}}{a_{1cr}} = -\frac{\mathrm{d}l}{2D}. \quad (36)$$

This is a very important relation for solving the gas dynamic problems of real gas flows in the presence of heat transfer.

With no heat transfer ($a_{1cr} = a_{cr}$) and when applied to an ideal gas ($\bar{\eta} = 1$; $\xi_{cr} = 1$; $k_T = k$; $\lambda = \lambda_{id}$), equation (36) undergoes integration and takes on the conventional form for gas dynamics

$$\frac{1}{\lambda_{1id}^2}-\frac{1}{\lambda_{2id}^2}-\ln\frac{\lambda_{2id}^2}{\lambda_{1id}^2} = \frac{2k}{k+1}\xi\frac{l}{D}. \quad (37)$$

Equation (36) cannot be integrated. It can only be solved for specific cases. With the aid of a computer, this equation can be used to determine the distribution of the variable critical velocity a_{cr} along a pipe, assuming in the first approximation that $\lambda = \lambda_{id}$. In the absence of heat transfer, this distribution is determined from relation (37). The validity of equation (36) will be justified in the ensuing paper.

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QUELQUES RELATIONS DE LA DYNAMIQUE DES GAZ POUR DES ECOULEMENTS DE GAZ REEL EN PRESENCE DE TRANSFERT THERMIQUE

Résumé—A partir des équations différentielles de la thermodynamique, quelques relations sont établies qui prennent en compte la non idéalité des gaz, le transfert d'énergie et le frottement.

EINIGE GASDYNAMISCHE BEZIEHUNGEN FÜR DIE STRÖMUNG REALER GASE BEI GLEICHZEITIGER WÄRMEÜBERTRAGUNG

Zusammenfassung—Auf der Grundlage thermodynamischer Differentialgleichungen werden einige Beziehungen entwickelt, die das nicht-ideale Verhalten des Gases sowie die Übertragung von Energie und Impuls berücksichtigen.

НЕКОТОРЫЕ ЗАВИСИМОСТИ ГАЗОДИНАМИКИ ДЛЯ ПОТОКОФ РЕАЛЬНЫХ ГАЗОВ ПРИ НАЛИЧИИ ТЕПЛООБМЕНА

Аннотация—На основе дифференциальных уравнений термодинамики выведены зависимости с учетом неидеальности, энергообмена и трения.